NORMAL BUNDLES & FULLY FAITHFUL FM-TRANSFORMS

ERIK NIKOLOV – DOGS-TALK 29/10

X, Y smooth projective varieties (over some field k) of dimensions $d_X \leq d_Y$.

Question: $D^b(X) \hookrightarrow D^b(Y)$?

Want to find a fully faithful triangulated functor $D^b(X) \xrightarrow{\Phi} D^b(Y)$, i.e. $\mathrm{Hom}^*_X(-,-) \xrightarrow{\sim} \mathrm{Hom}^*_Y(\Phi(-), \Phi(-))$. (automatically natural)

$\mathrm{Hom}^i_X(-,-) := \mathrm{Ext}^i_X(-,-)$
$:= \mathrm{Hom}_{D^b(X)}(-,(-)[i])$
$= H^i(R\mathcal{H}om^\bullet(-,-))$

Game plan:
1. Remarks on fully-faithfulness
2. First examples
3. A general criterion
4. Example: (Possibly singular) blow-ups

① Remarks (i) fully faithful $\Longleftrightarrow$ equivalence onto a full triangulated subcategory
($\Rightarrow$ essentially injective; $\not\Leftarrow$)

(ii) If Φ has both adjoints (✓): Can be checked on spanning classes like $\{\mathcal{O}_x\}_{x \in X \text{ closed}}$ or $\{\mathcal{L}^{\otimes i}\}_{i \in \mathbb{Z}}$, $\mathcal{L}$ ample line bundle on X.

$0 = (\ )^{\perp} = {}^{\perp}(\ )$

(iii) If $d_X < d_Y$, then $D^b(X) \xrightarrow{\Phi} D^b(Y)$ will not be an equivalence!

(Equivalences of derived categories preserve Serre-functors $\Rightarrow$ dimensions)

(iv) Y (smooth) projective $\Rightarrow$ $D^b(Y)$ saturated $\qquad$ ($D^{perf}(Y)$)

$\Rightarrow$ any full triang. subcategory is admissible

$\Rightarrow$ Φ induces semi-orthogonal decomposition

$$\langle \Phi(D^b(X)), \underbrace{{}^{\perp}\Phi(D^b(X))}_{\Phi(D^b(X)) \text{ adm.} \Rightarrow \text{ triangulated}} \rangle$$

<u>Thm</u> (Orlov) X, Y sm. proj., Φ fully faithful + triangulated

$\Rightarrow$ Φ is of Fourier-Mukai type
$$\begin{tikzcd} \mathcal{K} \overset{L}{\otimes} X \times Y \arrow[r] \arrow[d] & Y \\ X & \end{tikzcd}$$
! (limits choice of candidates)

<u>Today</u>: $\Phi = \iota_*(\mathcal{E} \otimes p^*(-))$ in the situation
$$\begin{tikzcd} V \arrow[r, hook, "\iota"] \arrow[d, "p"'] \arrow[dr, dashed, "\exists! \iota"] & Y \\ X & X \times Y \arrow[u] \arrow[l] \end{tikzcd}$$
where

p is smooth + proper, ι closed embedding, $\mathcal{E} \in Coh(V)$ loc. free.

<u>Note</u>: Φ is a Fourier-Mukai transform with kernel $\iota_* \mathcal{E}$!

(projection formula)

② <u>First examples</u> (i) $d_V = d_Y$, e.g. $V = \mathbb{P}(\mathcal{E}) \xrightarrow[\mathbb{P}^{r-1}]{p} X$ with $\mathcal{E}$ locally free:

(or iterated versions $\mathbb{P}(\mathcal{E}_n) \to \mathbb{P}(\mathcal{E}_{n-1}) \to \dots \to X$)

<u>Claim</u>: All the functors $p^*(-) \otimes \mathcal{O}_p(i)$ from Orlov's projective bundle formula are fully faithful!

($\mathcal{O}_p(i)$ = i-th relative twisting sheaf on $\mathbb{P}(\mathcal{E}) \to X$, On every fibre
$$\begin{tikzcd} \mathbb{P}^{r-1} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{P}(\mathcal{E}) \arrow[d] \\ \{x\} \arrow[r, hook] & X \end{tikzcd}$$
: $\mathcal{O}_p(i)|_{\mathbb{P}^{r-1}} = \mathcal{O}(1)^{\otimes i}$.)

<u>Proof</u>: Look at Hom-spaces:

$$\mathrm{Hom}^i_X(A,B) \xrightarrow{\Phi} \mathrm{Hom}^i_V(p^*A \otimes \cancel{\mathcal{O}_p(i)}, p^*B \otimes \cancel{\mathcal{O}_p(i)})$$

(twisting with line bundles = equivalence)

$$\cong \mathrm{Hom}^i_V(p^*A, p^*B) \underset{\text{(adj.)}}{\cong} \mathrm{Hom}^i_X(A, \underbrace{Rp_* p^* B}_{\cong Rp_*\mathcal{O}_{\mathbb{P}(\mathcal{E})} \overset{L}{\otimes} B})$$

Fully-faithfulness now easily follows from $Rp_*\mathcal{O}_{\mathbb{P}(\mathcal{E})} \cong \mathcal{O}_X[0]$ (use e.g. $\mathbb{P}(\mathcal{E}) \overset{\text{locally}}{\hookleftarrow} U \times \mathbb{P}^{r-1}$ + Künneth formula)!

(ii) $d_V = d_X$, i.e. $X \overset{\iota}{\hookrightarrow} Y$ closed embedding.

Non-derived $\iota_* : \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$ is fully faithful, but this becomes false for the derived $\iota_* : D^b(X) \to D^b(Y)$!

Closed embeddings are in general not fully faithful:

If $A = B = \mathcal{O}_X$, $H^i(\mathcal{O}_X) = 0$ for $i > 0$: (e.g. $X = \operatorname{Spec} k \hookrightarrow Y$)

$$0 = H^i(\mathcal{O}_X) = \operatorname{Hom}^i_X(\mathcal{O}_X, \mathcal{O}_X) \xrightarrow{\Phi} \underbrace{\operatorname{Hom}^i_Y(\iota_*\mathcal{O}_X, \iota_*\mathcal{O}_X)} \neq 0$$

Using $L\iota^* \dashv \iota_* \dashv \iota^!$: will be related to cohomology of normal bundle & wedge powers $\Lambda^\bullet N_{X/Y}$! Need not vanish.

Think of $\operatorname{Ext}^i(\iota_*\mathcal{O}_x, \iota_*\mathcal{O}_x) = \Lambda^i T_x Y \neq 0$

(iii) $d_X = 0$, e.g. $V \overset{\iota}{\hookrightarrow} Y$:
$$\begin{array}{c} V \\ \downarrow \\ \operatorname{Spec} k \end{array}$$

$\Phi = \iota_* p^*$ fully faithful

$\Updownarrow$ (Test on $k[0] \in D^b(\operatorname{Spec} k)$)

$\iota_*\mathcal{O}_V$ is an exceptional object in $D^b(Y)$

$\Updownarrow$

$\operatorname{Hom}^*_Y(\iota_*\mathcal{O}_V, \iota_*\mathcal{O}_V) = k[0]$

<u>Claim</u> In the case of a sm. curve $V = C$ in a surface $Y = S$,

$\iota_*\mathcal{O}_C$ exceptional in $D^b(Y) \iff C$ rational curve, $C^2 = -1$.

($\iff \Phi$ fully faithful) (exceptional curve of first kind)

Hence on a minimal surface, there are no exceptional objects of the form $\iota_*\mathcal{O}_C$, $C \subseteq S$ curve.

<u>Proof</u> $\mathrm{Hom}^i_S(\iota_*\mathcal{O}_C, \iota_*\mathcal{O}_C) \cong \mathrm{Hom}^i_C(\underbrace{L\iota^*\iota_*\mathcal{O}_C}_{\text{complex, not only sheaf}}, \mathcal{O}_C)$

$$E_2^{p,q} = \mathrm{Hom}^p(L^q\iota^*\iota_*\mathcal{O}_C, \mathcal{O}_C) \Rightarrow$$ (spectral sequence)

Since $C \overset{\iota}{\hookrightarrow} S$ is Cartier $\Rightarrow$ get loc. free resolution

$$0 \to [\mathcal{O}_S(-C) \to \mathcal{O}_S] \to \iota_*\mathcal{O}_C \to 0$$

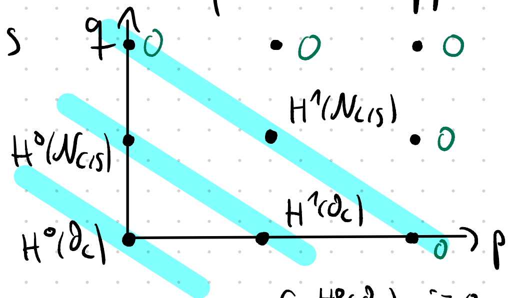
$$\Rightarrow L\iota^*\iota_*\mathcal{O}_C = \iota^*[\mathcal{O}_S(-C) \to \mathcal{O}_S]$$
$$= [\underbrace{\mathcal{O}_C(-C)}_{\mathcal{N}^\vee_{C/S}} \xrightarrow{0} \mathcal{O}_C]$$

$$\Rightarrow L^q\iota^*\iota_*\mathcal{O}_C = \begin{cases} \mathcal{O}_C, & q = 0 \\ \mathcal{N}^\vee_{C/S}, & q = 1 \\ 0, & q \geq 2. \end{cases}$$

In general: $C = V$ reduced lci inside Y smooth
$\Rightarrow$ locally, $V = V(s)$, $s \in H^0(\mathcal{E})$ regular section,
Koszul complex: resolution
$0 \to \Lambda^c\mathcal{E}^\vee \to \dots \to \mathcal{E}^\vee \to \mathcal{O}_Y \to \iota_*\mathcal{O}_V \to 0$,
$\mathcal{N}_{V/Y} \cong \mathcal{E}|_V$, $\mathcal{E} = \mathcal{O}_S(C)$ here!

In general: $L^q\iota^*\iota_*\mathcal{O}_V \cong \Lambda^{-q}\mathcal{N}^\vee_{V/Y}$
(Huybrechts FM-Trafos §11)

$\Rightarrow$ The spectral sequence approximating $\mathrm{Hom}_S^{i=p+q}(\iota_*\mathcal{O}_C, \iota_*\mathcal{O}_C)$
becomes



(dim $C = 1$)

$$\Rightarrow \mathrm{Hom}^i_S(\iota_*\mathcal{O}_C, \iota_*\mathcal{O}_C) = \begin{cases} H^0(\mathcal{O}_C), & i=0 \\ H^0(\mathcal{N}) \oplus H^1(\mathcal{O}_C), & i=1 \\ H^1(\mathcal{N}), & i=2 \\ 0, & i \geq 3 \end{cases} \underset{\text{(except. / fully faithful)}}{\overset{!}{=}} \begin{cases} k, & i=0 \\ 0, & i \geq 1 \end{cases}$$

Now $h^0(\mathcal{O}_C) = 1 \Rightarrow C$ connected, $h^1(\mathcal{O}_C) = g(C) = 0 \Rightarrow C \cong \mathbb{P}^1$,
$\mathcal{N}_{C/S} = \mathcal{O}(i)$, $i = \deg \mathcal{N}_{C/S} = C^2$, $H^0(\mathcal{N}_{C/S}) = 0 \Rightarrow \boxed{i = -1}$. //

Cor For Blow-Ups of smooth surfaces in a point, the exceptional divisor is an exceptional object.

③ **Main Prop.** Consider a correspondence $\begin{array}{ccc} V & \overset{\iota}{\hookrightarrow} & Y \\ \downarrow p & & \\ X & & \end{array}$ as before, and a

(A general criterion)

FM-Trafo $\Phi = \iota_*(\mathcal{E} \otimes p^*(-))$, with $\mathcal{E}$ locally free.

Then Φ is fully faithful, provided that on every fibre of p:

- $\mathrm{Hom}_{V_x}(E, E) \cong k \cdot \mathrm{id}$, $E = \mathcal{E}|_{V_x}$,
- $H^p(V_x, \underbrace{\Lambda^q(\mathcal{N}_{V/Y})|_{V_x}}_{\text{only exists if } V \subsetneq Y} \otimes \underbrace{E \otimes E^\vee}_{\text{vanishes for } E = \text{line bundle}}) = 0 \quad p+q>0$

$$\begin{tikzcd} V_x \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] \arrow[rr, bend left, "j"] & V \arrow[r, hook, "\iota"] \arrow[d, "p"] & Y \\ \{x\} \arrow[r, hook] & X & \end{tikzcd}$$

Proof ingredients

- Use Bondal–Orlov-criterion to prove fully-faithfulness of Φ on spanning class $\{\mathcal{O}_x\}_{x \in X}$ only in certain degrees
- $\mathrm{Hom}^i_Y(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x)) \cong \mathrm{Hom}^i_Y(j_*E, j_*E)$, use $j_* \dashv j^!$, approximate by spectral sequence.
- $\mathcal{H}^q(j^! j_* E) \cong E \otimes \mathcal{H}^q(j^! j_* \mathcal{O}_{V_x}) \cong E \otimes \mathcal{H}^{q-c}(Lj^* j_* \mathcal{O}_{V_x}) \otimes \det \mathcal{N}_{V_x/Y}$

  $\cong \Lambda^{c-q} \mathcal{N}_{V_x/Y}$, use s.e.s. $0 \to \mathcal{O}_{V_x}^{\oplus d} \to \mathcal{N}_{V_x/Y} \to \mathcal{N}_{V/Y}|_{V_x} \to 0$

see previous page, also works for lci's!

Remarks (i) There is a similar criterion for semi-orthogonality, see my arXiv-preprint.

(ii) It is not clear to me why the converse of the statement should hold. Let me know if you have a proof in mind!

4 **Example** Consider a blow-up of X (potentially singular) in a smooth center Y. Then we have a cartesian square

$$\begin{array}{ccc} E_YX & \overset{\iota}{\hookrightarrow} & \tilde{X} \\ p\downarrow & \square & \downarrow \\ Y & \hookrightarrow & X \end{array}$$

and can consider $\Phi = \iota_*(\mathcal{L} \otimes p^*(-))$, $\mathcal{L}$ line bundle.

$E_YX = \mathrm{Proj}_Y \bigoplus_d \mathcal{J}^d/\mathcal{J}^{d+1}$
$\mathcal{J} = \mathcal{J}_{Y/X}$

Assume that $E_YX \to Y$ has fibres all isomorphic to $\mathbb{P}^{c-1}$, so that $\mathcal{O}_p(-k) = \mathcal{N}_{E_YX/\tilde{X}}$, $k \geq 0$.

(If $Y \overset{lci}{\subseteq} X$: $\mathcal{N}_{Y/X}^\vee$ locally free, $\mathrm{Sym}^* \mathcal{J}/\mathcal{J}^2 = \bigoplus_d \mathcal{J}^d/\mathcal{J}^{d+1}$, $E_YX \cong \mathbb{P}(\mathcal{N}_{Y/X}^\vee)$, $\mathbb{P}^{c-1}$-bundle with $c = \mathrm{codim}(Y,X)$)

Then Φ is fully faithful if $c-1 \geq k \geq 1$.

Remark: Part of Orlov's blow-up formula where X, Y are smooth, $k=1$, $c = \operatorname{codim}(Y, X) \geq 2$.

Proof of fully faithfulness: Look at fibres + the two conditions:

- $\operatorname{Hom}_{\mathbb{P}^{c-1}}(\mathcal{L}_{|\mathbb{P}^{c-1}}, \mathcal{L}_{|\mathbb{P}^{c-1}}) = H^0(\mathbb{P}^{c-1}, \mathcal{O}) = k$ ✓
- $H^p(V_x, \Lambda^q(\mathcal{N}_{V/Y})_{|V_x} \otimes E \otimes E^\vee) = H^p(\mathbb{P}^{c-1}, \Lambda^q \mathcal{O}(-k)) = ?$

  $q=0$: $H^*(\mathbb{P}^{c-1}, \mathcal{O}) = k[0]$ (see above)

  $q=1$: $H^*(\mathbb{P}^{c-1}, \mathcal{O}(-k))$ at most concentrated in degrees 0 or $c-1$.
  Vanishes if $-k<0$ or $-k \geq -(c-1)$. ✓

Example: Hilbert scheme $X^{[2]}$ of two points, blowup of $X^{(2)} = X^2/S_2$:

$$\begin{tikzcd} \mathbb{P}(\Omega_X) \arrow[r, hook, "\iota"] \arrow[d, "p"] \arrow[dr, phantom, "\square"] & X^{[2]} \arrow[d, "\text{blow-up}"] \\ X \arrow[r, hook, "\Delta"'] & X^{(2)} \end{tikzcd}$$

$\mathcal{N}_{\mathbb{P}(\Omega_X)/X^{[2]}} \cong \mathcal{O}_p(-2)$

$\Rightarrow$ $D^b(X) \hookrightarrow D^b(X^{[2]})$ if $\dim X \geq 3$.

(More such functors are known, see work of Krug, Ploog, Sosna)